

# Scenarios and their Aggregation in the Regulatory Risk Measurement Environment

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## Abstract

We define scenarios, propose different methods of aggregating them, discuss their properties and benchmark them against quadrant requirements.

*Keywords:* quadrant requirements, available capital, scenarios, scenario aggregation, risk measurement

## 1 Terminology

We assume that the *available capital* of an insurance company can be described as a function of observable real variables, called *risk factors*.

Examples of risk factors are basic economic variables such as interest rate (yield curve), share index, real estate, FX (foreign exchange) and corporate spreads.

The dependency of the available capital on these risk factors is described by the *valuation function*  $V$  of an insurance company:

$$V : \mathbb{R}^n \rightarrow \mathbb{R}$$

where

$$(x_1, \dots, x_n) \mapsto V(x_1, \dots, x_n).$$

We require  $V$  to be *measurable* with respect to the Borel  $\sigma$  algebras  $(\mathfrak{B}_n, \mathfrak{B})$ , and we consider  $\mathbb{R}^n$  together with a probability measure  $P$  with

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respect to the Borel  $\sigma$  algebra  $\mathfrak{B}_n$ . This measure can be thought of as representing the distribution of the risk factors at a given future point in time. Define  $P_V = V_*P$  as image measure on  $\mathbb{R}$ ; this describes the distribution of the risk bearing capital. Sometimes we use the notion *risk bearing capital* as synonymous to the one of available capital.

## 2 Risk measurement vs risk measurement for regulatory purposes

Risk measurement requires the following steps to be performed:

1. Identifying a suitable set of risk factors
2. Making an assumption about the distribution of risk factors (i.e. choosing  $P$ )
3. Describing the dependency of available capital on the risk factors (i.e. defining  $V$ )
4. Analysing  $P_V$

Steps 3. and 4. are clearly entity specific. In a regulatory environment, steps 1. and 2. may be subject to minimum requirements from the supervisor to guard against lack of awareness or misjudgements by the company's management.

Regarding step 1., there are basic economic variables like interest rates, which may have a major impact on most entities. Therefore supervisors require them to be considered. Additional risk factors, which are important only to a specific entity, may be included in the analysis by the affected entities (e.g. exotic asset classes).

Regarding step 2., for purposes of equal treatment of all supervised entities, it is important to note that the distribution of the “basic” risk factors prescribed by the supervisor for regulatory purposes is identical for all supervised entities in one jurisdiction, although there may be different views on these distributions.

For example, interest rates in one year's time will be the same for all entities, although they may affect different entities to a different degree. So, the distribution of the risk factor interest rate should be the same for all entities; what is different, though, is how the individual  $V$  acts on that distribution (step 3). In short, whilst  $P$  should be driven by (minimum)

requirements from the supervisor,  $P_V$  is individually determined based on  $V$ .

In a framework for determining *economic capital*, which is a fully internal exercise, steps 1. to 4. are clearly *not* subject to regulatory requirements. Sometimes, this leads to a misunderstanding, which we formulate in

**Fallacy 1.** We (company management) have our own view on the interest rates in one year's time which we use as well for our risk management and economic capital purposes, and we want to use it for the regulatory capital requirements as well.

Following such an approach in general would lead to regulators being unable to prescribe the estimation of the future behaviour of risk factors. The fallacy arises because, by definition, regulators set regulatory requirements.

### 3 Mathematically elegant way of setting requirements on risk factors

Usually, supervisors are interested in the behaviour of entities under extreme events. In this section, we introduce a very simple toolbox which can be used to construct and describe extreme events.

**Definition 1.** For each non-zero linear form  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$ , a set of the form  $\lambda^{-1}([c, \infty))$  is called an *affine half-space*. A *quadrant* is a non-empty intersection of a finite set of affine half-spaces.

By definition, every quadrant is an element of  $\mathfrak{B}_n$ , and  $\mathfrak{B}_n$  is generated by the set of all quadrants.

*Example 2.* Any affine subspace of  $\mathbb{R}^n$  is a quadrant. In particular, a subset consisting of one point only is a quadrant.

**Definition 3.** Let  $A_Q$  be a quadrant and  $p_Q \in [0, 1]$ . A *quadrant requirement* is a couple  $Q = (A_Q, p_Q)$ .  $P$  is said to *fulfill the quadrant requirement*  $Q$  if it satisfies the following condition

$$P(A_Q) \geq p_Q$$

*Remark 4.* Even if it would be possible to formulate requirements on more general sets than quadrants, we feel that the setting chosen above will be sufficient for regulatory purposes.

In setting quadrant requirements, supervisors can express their judgement on the future behaviour of risk factors. They can ensure that the measures  $P$  used by the entities have sufficient weight in the tail of the common distribution of the risk factors. Suppose, for example, a supervisor believes that the following statement is relevant for risk measurement for regulatory purposes:

“In one year’s time, the interest rate of the ten year’s Swiss frank is less or equal to 0.5% with a probability of 1%.”

He could translate this statement to the following quadrant requirement:

$$P(i_{10} \leq .5\%) \geq 1\%,$$

where the variable  $i_{10}$  describes the return rate of the ten year’s Swiss frank. This quadrant requirement should be the same and equal for all entities under supervision, what is different is the impact of that requirement on the individual insurer’s available capital. Suppose a Company 1 which is completely hedged against movements of the ten year Swiss rate, thus  $\Delta V_1(i_{10}) = 0$ , where  $\Delta V$  describes the change of the available capital. A Company 2, which refrains from hedging, is likely to have the result  $\Delta V_2(i_{10}) < 0$ .

**Definition 5.** A set of quadrant requirements  $M$  is an arbitrary finite set of quadrant requirements such that the number  $p_M = \sum_{Q \in M} p_Q \leq 1$ .

A set of quadrant requirements is useful for a supervisor to set his requirements on more than one risk factor, and helpful to formalize his judgement on tail dependencies. This does not mean that all companies in one jurisdiction have to use the same  $P$ , it only means that the set of acceptable  $P$  is restricted by the supervisor.

Quadrant requirements have also the appealing property that it is easy to check in an objective and reproducible manner whether or not they are fulfilled. Indeed, subjective judgement is reduced to such an extent that an independent third party or law court could easily double check the supervisor’s assessment and come to the same result.

There might be as well other regulatory requirements on  $P$  such as e.g. being “realistic”, “state of the art”, and other “qualitative” criteria. These are very important to gain a better mutual understanding of  $P$  and should be intensively discussed in the regulatory dialogue between the company and the supervisor. The constant challenge with these “qualitative” criteria is that, due to their inherent element of subjectivity, an independent third

party or law court does not necessarily need to come to the same result as the supervisor. A way out could be transforming the rather subjective “qualitative” criteria into objectively testable quadrant requirements.

We conclude the section with quantifying the statement that “a supervisor should neither be substantially under- nor over-prescriptive” in terms of the number of quadrant requirements to be used. Suppose a supervisor wants one and only one  $P$  to be used in his jurisdiction. How many quadrant requirements would he need? An answer is given by:

**Theorem 1.** *Let  $P$  be any probability measure on  $(\mathbb{R}^n, \mathfrak{B}_n)$ . Then there exists a countable set of pairs  $(A_i, p_i)$  of quadrants  $A_i$  and real numbers  $p_i \in [0, 1]$  such that  $P$  and only  $P$  fulfills all quadrant requirements  $(A_i, p_i)$ .*

*Proof.* Consider hypercubes whose corners are in  $\mathbb{Q}^n$ . These hypercubes are quadrants, and the set of these hypercubes is countable. Let  $A_i$  be an enumeration of these hypercubes. Define  $p_i$  by  $p_i = P(A_i)$ . Then by definition of  $p_i$ ,  $P$  satisfies all quadrant requirements  $(A_i, p_i)$ .

Conversely, assume  $P'$  satisfies all quadrant requirements  $(A_i, p_i)$ . Consider the set  $\mathfrak{M}$  of all  $X \in \mathfrak{B}_n$  for which  $P(X) = P'(X)$ . Then  $\mathfrak{M}$  is a  $\sigma$  algebra. Also,  $A_i \in \mathfrak{M}$ . Indeed,  $P'(A_i) \geq P(A_i)$  by definition of the quadrant requirements. On the other hand,  $\mathbb{R}^n \setminus A_i$  can be written as a union of at most countably many disjoint hypercubes  $A_j$  where  $j \in J(i)$ , hence

$$\begin{aligned} 1 - P'(A_i) &= P'(\mathbb{R}^n \setminus A_i) = \sum_{j \in J(i)} P'(A_j) \\ &\geq \sum_{j \in J(i)} P(A_j) = P(\mathbb{R}^n \setminus A_i) = 1 - P(A_i) \end{aligned}$$

This shows that  $P'(A_i) \leq P(A_i)$ , so we have  $A_i \in \mathfrak{M}$ .

As  $\mathfrak{B}_n$  is generated by  $A_i$ , we have  $\mathfrak{M} = \mathfrak{B}_n$ . Thus,  $P = P'$ . □

Now we can classify as follows:

- *No quadrant requirements at all* is equivalent to *The distribution  $P$  is exclusively the choice of each supervised entity* (no regulatory prescription at all, cf. Fallacy 1).
- *Countably many requirements on quadrants* can be defined so that *the distribution  $P$  is determined exclusively by the supervisor* (supervisor could well be regarded as “overly prescriptive”).

## 4 Scenarios and case study: use of scenarios in the SST

The use of scenarios and stress testing is currently much discussed within the regulatory community. We give a general definition of scenarios and briefly describe their use under the Swiss Solvency Test (SST).

**Definition 6.** A scenario  $s$  is an element  $d_s \in \mathbb{R}^n$ , and an impact of the scenario  $s$  is the value  $V(d_s)$ ;  $d_s$  is also called *stressed situation*.

$d_s$  can be thought of as a concrete realization of the underlying risk factors. It is a powerful tool in the “what-if-analysis”, to answer the question:

”What happens to Company 1 under a certain change of risk factors?”

In order to do so, the supervisor might specify  $d_s$ , the “if”, the stressed situation, and Company 1 evaluates the impact of the scenario  $d_s$ , gives the “what” by calculating  $V_1(d_s)$ . This is “qualitative” in the sense that the impact of scenarios is discussed between the entity and the supervisor. The impact of a scenario might or might not increase the regulatory prescribed capital required (PCR). Switzerland is a jurisdiction where scenarios impact the PCR, and we briefly describe how this is done under the current Swiss regime, the SST:

**Definition 7.** An *enhanced scenario*  $S$  is a couple  $S = (d_S, p_S)$  consisting of  $d_S \in \mathbb{R}^n$  and  $p_S \in [0, 1]$ , where  $p_S$  is called probability of occurrence of  $S$ ,  $d_S$  is called *deflection* of  $S$ .

**Definition 8.** A scenario set  $M$  is an arbitrary finite set of enhanced scenarios such that  $p_M = \sum_{S \in M} p_S \leq 1$ .

The scenario is enhanced in the sense that it comes with a probability of occurrence, which is set by Swiss supervisors. It is needed to calculate the *target capital*, as the PCR is referred to in the SST, and usually more than one scenario is used. In Switzerland, the target capital is determined via a distribution-based approach. The way of taking scenarios into account in the target capital is to modify the initially assumed distribution in a certain way, which is usually referred to as *aggregation*.

In the following way Swiss supervisors currently aggregate the impact of the scenarios. Strictly speaking, this method is only prescribed within the SST standard model, but roughly speaking, it currently constitutes as

well a sort of industry standard in the Swiss insurance sector, even for some entities using internal models for regulatory purposes.

**Definition 9** (SST scenario aggregation). Let  $M$  be a scenario set,  $P$  a probability measure on  $(\mathbb{R}^n, \mathfrak{B}_n)$  and  $V$  a valuation function. Consider the expression

$$(1 - p_M)V_*P + \sum_{S \in M} p_S(\tau_{V(d_S)}V_*P),$$

where  $\tau_{V(d)}$  denotes the translation in  $\mathbb{R}$  in by  $V(d)$ . This expression is called *SST scenario aggregation*.

Thus, the SST scenario aggregation is a *mixture* of the starting distribution  $V_*P$  (describing the distribution of risk bearing capital *before* aggregation of scenarios) with its translated versions, where the translation is given by the impact of each scenario. However, the fact that the valuation function  $V$  gets involved during scenario aggregation obscures the view on what is happening. As the scenarios themselves are defined at the risk factor level, it seems desirable to have a definition of what scenario aggregation means at this level. This might be achieved by the following alternative approach:

**Definition 10** (aggregation by shifting). Let  $M$  be a scenario set,  $P$  a probability measure on  $(\mathbb{R}^n, \mathfrak{B}_n)$ . Define

$$P_M = (1 - p_M)P + \sum_{S \in M} p_S \tau_{d_S}P$$

where  $\tau_d$  denotes  $v \mapsto v + d$ , the *translation* by  $d$  in  $\mathbb{R}^n$ . We say that  $P_M$  is obtained from  $P$  by aggregating enhanced scenarios from scenario set  $M$  to  $P$ .

Thus, similar to the situation above, the resulting distribution of the risk factors is a *mixture* of the starting distribution  $P$  with its translated versions, where the translation is given by the deflection of each scenario. So one might ask whether the SST scenario aggregation at the level of the risk bearing capital leads to the same result as the aggregation by shifting on the risk factor level defined above. The following lemma deals with this question.

**Lemma 1.** *The aggregation of enhanced scenarios acts on the distribution of risk bearing capital in a similar way as on the distribution of risk factors: the mixture of the distribution of the underlying risk factors carries over to the distribution of risk bearing capital, where the translation by the*

deflection of the scenarios is being replaced by a translation by the impact of the scenarios, and the valuation function  $V$  is replaced by its twisted versions  $V_d$ .

The twisted valuation functions are defined by

$$V_d(x) := V(x + d) - V(d)$$

If  $V$  is additive, then  $V_d$  coincides with its twisted versions, and in this case SST scenario aggregation leads to the same result as the aggregation by shifting on the risk factor level.

*Proof.* Note that

$$\begin{aligned} V_*P_M &= V_*((1 - p_M)P + \sum_{S \in M} p_S \tau_{d_S} P) \\ &= (1 - p_M)V_*P + \sum_{S \in M} p_S V_*(\tau_{d_S} P) \\ &= (1 - p_M)V_*P + \sum_{S \in M} p_S (\tau_{V(d_S)_*} V_{d_S} P). \end{aligned}$$

In the last step, we have used the fact that

$$V \circ \tau_d = \tau_{V(d)} \circ V_d.$$

If  $V$  is additive, then clearly  $V_d = V$  and the expression obtained above coincides with the definition of SST scenario aggregation.  $\square$

So, for additive  $V$  the aggregation by shifting carries over to the SST scenario aggregation. However, in risk measurement  $V$  is generally not additive.

One further challenge with aggregation by shifting is, that the company specific distribution  $P$  serves as input. Companies evaluate the difference between their starting  $P$  and the  $P_M$  after scenario aggregation to understand the impact of the scenarios. If a starting  $P_1$  from Company 1 in terms of PCR is “harder” than  $P_2$  of Company 2, this carries over to the distributions  $P_{1M}$  and  $P_{2M}$ , if both companies are asked by the supervisor to aggregate the same scenario set  $M$ . So it could be regarded to be fairer, to allow for different scenarios sets  $M_1$  and  $M_2$ , which reflects in some way the differences between the starting distributions  $P_1$  and  $P_2$ . In such an environment, it is very difficult for the supervisor to evidence equal treatment. Furthermore, with many companies in one jurisdiction, it is not very



practicable for the supervisors to prescribe a different scenario set for all the companies.

In the following sections, it is our program to develop criteria which overcome these challenges using the quadrant language from section 3.

## 5 Translating quadrant requirements into the scenario language

In this section, our aim is to show that if a supervisor would like to impose quadrant requirements, he can achieve the same aim by requiring the aggregation of scenarios.

Unfortunately, if we use aggregation by shifting, this statement only holds in a weaker form and under additional assumptions on the nature of the quadrant requirements. The deeper reason behind this is that the mechanics of aggregation by shifting strongly depends on the initially assumed distribution  $P$ .

Therefore, we initially introduce a more convenient, alternative aggregation method, which we call “point mass aggregation”. This helps us establishing our statement in a more general context.

Only then we return to the SST scenario aggregation and investigate, which additional assumptions are necessary to establish our statement when aggregation by shifting is applied by the supervisor.

Basically, this means that a supervisor relying on aggregation by shifting acts self-restrictive in terms of the quadrant requirements he is able to impose.

**Definition 11** (point mass aggregation). Let  $M$  be a scenario set,  $P$  a probability measure on  $(\mathbb{R}^n, \mathfrak{B}_n)$ . Define

$$P_M^{pt} = (1 - p_M)P + \sum_{S \in M} p_S \delta_{d_S}$$

where  $\delta_d$  denotes the Dirac measure centered at  $d$ . We say that  $P_M^{pt}$  is obtained from  $P$  by aggregating enhanced scenarios from scenario set  $M$  as point-mass to  $P$ .

The name of this aggregation method should be intuitive: It means, that scenarios are aggregated by adding point masses at each scenario deflection with the corresponding probability; on the other hand, the probability weight of the initial distribution  $P$  is reduced accordingly, so that the resulting measure is a probability measure.

**Theorem 2.** Let  $N = \{Q_1, Q_2, \dots, Q_k\}$  be a set of quadrant requirements. Then there exists a scenario set  $M$  such that for any distribution  $P$ , the distribution  $P_M^{pt}$  resulting from aggregating the enhanced scenarios from  $M$  as point-mass to  $P$  satisfies the quadrant requirements from  $N$ .

*Proof.* Denote the quadrants and probabilities of the quadrant requirements by  $A_j$  and  $p_j$ , respectively. As  $A_j \neq \emptyset$ , we can choose  $d_j \in A_j$ .  $S_j = (d_j, p_j)$  defines an enhanced scenario, and we define  $M = \{S_1, S_2, \dots, S_k\}$ . Then for each  $j$ ,  $P_M^{pt}$  has a point-mass of weight  $p_j$  at  $d_j$  by definition of point-mass aggregation. As  $A_j \supseteq \{d_j\}$ ,  $P_M^{pt}(A_j) \geq P_M^{pt}(\{d_j\}) \geq p_j$ . This shows that  $P_M^{pt}$  satisfies the quadrant requirements from  $N$ .  $\square$

Note that the theorem does not hold if we replace point mass aggregation by aggregation by shifting, as can be seen from the following

*Example 12.* Let  $p_{\max} \in [0, 1]$  and  $P$  such that  $P(B) < p_{\max}$  for all balls  $B$  with fixed radius  $R \geq 0$ . For any quadrant  $A$  contained in such a ball  $B$  with radius  $R$ ,  $P$  never fulfills the quadrant requirement  $(A, p_{\max})$ . Additionally, for any scenario set  $M$ ,  $P_M$  does not satisfy the quadrant requirement  $(A, p_{\max})$ . Indeed, aggregation by shifting is based on translations, and the radius of any ball is invariant under translations. Thus for any ball  $B$  we have

$$\begin{aligned} P_M(B) &= (1 - p_M)P(B) + \sum_{S \in M} p_S \tau_{d_S*} P(B) \\ &= (1 - p_M)P(B) + \sum_{S \in M} p_S P(\tau_{-d_S}(B)) \\ &< (1 - p_M)p_{\max} + \sum_{S \in M} p_S p_{\max} = p_{\max} \end{aligned}$$

Hence  $P_M$  also has the property that  $P(B) < p_{\max}$  for all balls  $B$  with fixed radius  $R \geq 0$ , so the assertion follows.

The situation above is easily achieved, as we show in the following

*Example 13.* Assume that  $P$  is a distribution with density with respect to the Lebesgue measure.

Assume further that we have a set of quadrant requirements, such that all quadrants are bounded and contained in a ball of radius  $R \geq 0$ . Define  $p_{\max} = \max_j(p_j) > 0$ , where  $p_j$  are the probabilities associated to the quadrant requirements, and denote the quadrant associated to the maximum by  $A$ .

Let  $\text{Mult}_c$  denote multiplication by  $c \in \mathbb{R}$  within  $\mathbb{R}^n$ . Then, we first observe that there exists  $c \in \mathbb{R}$  such that the image measure  $P' = \text{Mult}_{c*}P$  has the property that  $P'(B) < p_{\max}$  for *any* ball  $B$  of radius  $R$ .

Indeed, we can find a suitably countable partition of  $\mathbb{R}^n = \dot{\cup}_{i \in I} A_i$  such that

- $P(A_i) < \frac{p_{\max}}{2^n} \forall i$
- all  $A_i$  are contained in balls of radius  $|r| > 0$
- for any ball  $B$  with radius  $|r| > 0$  there is  $J \subseteq I$  with  $|J| \leq 2^n$  and  $B \subseteq \dot{\cup}_{j \in J} A_j$

We can then choose  $c = R/r$ , and it is easily verified that this  $c$  has the required property from example 12.

Astonishingly, even for an easy distribution family as the normal, it is not always possible to find a scenario set if we use the aggregation by shifting.

We thus found a contradiction to following

**Fallacy 2.** One can always correct  $P$  by aggregating scenarios by shifting in a way that prescribed quadrant requirements are satisfied.

However, under stronger assumptions on the quadrant requirements, we are able to establish a weaker result also for aggregation by shifting.

**Definition 14.** A convex set is said to be two-sided constrained, if it is contained in a suitable set of the form  $\lambda^{-1}([a, b])$ , where  $0 \neq \lambda$  is a linear form on  $\mathbb{R}^n$  and  $a, b \in \mathbb{R}$ .

A quadrant being two-sided constrained basically means, that it is defined by two counteracting conditions. An example could be a quadrant defined by the 10-year-rate being in the range between 0.5% and 1%. In the regulatory context, especially in cases where it is known which direction of move of a risk factor is adverse, it is more common not to consider quadrants which are two-sided constrained, as one does not wish to exclude “more extreme” situations from the consideration; however, it may still be useful to use quadrants which are two-sided constrained in some cases. As an example, by buying derivatives a company might change its risk profile such that the valuation function  $V$  is no longer monotonic in a certain risk factor, but known to have a minimum in an interval  $[a, b]$ . In this situation, it might be useful to consider events from a quadrant which is two-sided constrained.

We briefly discuss what definition 14 means for the real numbers in

*Example 15.* Consider a connected  $\emptyset \neq S \subseteq \mathbb{R}$ . If  $-\infty < \inf(S)$  and  $\sup(S) < \infty$ , then  $S \subset [\inf(S) - 1, \sup(S) + 1]$  is two-sided constrained. If  $\inf(S) = -\infty$  or  $\sup(S) = \infty$ , then  $S$  contains intervals of arbitrary length. Thus, a convex  $S \subseteq \mathbb{R}$  is either two-sided constrained or it contains line segments of arbitrary length.

We can now come to

**Theorem 3.** *Let  $Q$  be a set of quadrant requirements satisfying the following two additional conditions*

1. *None of the quadrants of  $Q$  is two-sided constrained*
2. *The cumulative probability satisfies  $p_Q < 1$*

*Then for any distribution  $P$ , there exists a scenario set  $M$  such that the distribution  $P_M$  satisfies the quadrant requirement from  $Q$ .*

*Remark 16.* Note that this result is much weaker than Theorem 2, as the scenario set chosen depends on  $P$ .

For the proof of Theorem 3 we will need some properties of quadrants, which we formulate in a more general setting for convex sets as

**Lemma 2.** *Let  $S \subseteq \mathbb{R}^n$  be a convex set. Then exactly one of the following two statements holds*

1.  *$S$  is two-sided constrained.*
2.  *$S$  contains  $n$ -dimensional balls of arbitrarily large radius as subsets.*

The proof can be found in the appendix.

*Remark 17.* We have not specified the norm with respect to which we consider the balls in the lemma. However, if  $S$  contains arbitrarily large balls with respect to one norm, it will contain arbitrarily large balls with respect to any norm, as all norms on  $\mathbb{R}^n$  are equivalent.

The idea of the proof of Theorem 3 can now be briefly outlined as follows: Given  $P$ , by its  $\sigma$ -additivity, we can find a large ball of radius  $R$  such that “most” of the mass of  $P$  lies inside this ball. By Condition 1 and Lemma 2 we can shift  $P$  by a “deflection” such that this ball lies inside a given quadrant. By taking this deflection to define a scenario, we cannot achieve that “all” the mass lies inside the corresponding quadrant, but most of it. By then increasing the probabilities associated to the quadrants only slightly such

that the sum remains below 1 - which can be achieved due to Condition 2 of the theorem - we can ensure that sufficient mass lies inside each quadrant, and therefore the quadrant requirement will be satisfied.

*Proof.* Denote the quadrant requirements from  $Q$  by  $(A_i, p_i)$ , and let  $N$  be the number of quadrants. As  $p_Q < 1$ , we can choose  $\epsilon > 0$  such that  $\frac{p_Q}{1-\epsilon} < 1$ . Because of  $\sigma$  additivity and  $\mathbb{R}^n = \cup_{R \in \mathbb{N}} B_R$ , we have  $\lim_{R \rightarrow \infty} P(B_R) = 1$ , where  $B_R$  denotes a ball of radius  $R$  centered at 0. Hence, we can choose  $R$  such that  $P(\mathbb{R}^n \setminus B_R) < \epsilon$ .

By Condition 1 and Lemma 2 we can find a deflection  $d_i$  for  $i = 1 \dots N$ , such that the ball of radius  $R$  centered at  $d_i$  lies inside  $A_i$ .

Take  $M$  to be the set of enhanced scenarios  $(d_i, p'_i)$ , where  $p'_i = \frac{p_i}{1-\epsilon}$ . We claim that  $P_M$  satisfies the quadrant requirements from  $Q$ , as desired. Indeed, for any  $A_i$  we have

$$\begin{aligned} P_M(A_i) &= (1 - p_M)P(A_i) + \sum_{S \in M} p'_S \tau_{d_S} P(A_i) \\ &= (1 - p_M)P(A_i) + \sum_{S \in M} p'_S P(\tau_{-d_S}(A_i)) \\ &\geq p'_i P(\tau_{-d_{S_i}}(A_i)) \geq p'_i P(B_R) > p'_i(1 - \epsilon) = p_i \end{aligned}$$

□

Fortunately, the concept of point mass aggregation has the appealing property that the converse of Theorem 2 holds as well:

**Theorem 4.** *Let  $M$  be a scenario set. Then there exists a set  $N$  of quadrant requirements, such that any  $P$  fulfilling the quadrant requirements from  $N$  can be written in the form  $P = P_M^{pt}$  with a suitable probability measure  $P'$ .*

*Proof.* Denote the enhanced scenarios from  $M$  by  $(d_1, p_1), (d_2, p_2), \dots, (d_m, p_m)$ . Define quadrants by  $Q_j = \{d_j\}$ . Let  $N$  be the set of quadrant requirements defined by  $Q_j$  and the associated probabilities  $p_j$ . We claim that  $N$  has the desired property. Indeed, if  $P$  is a probability measure fulfilling the quadrant requirements from  $N$ , in case  $p_M < 1$  we can define

$$P' = \frac{1}{1 - p_M} \left( P - \sum_{S \in M} p_S \delta_{d_S} \right).$$

It is easily verified that  $P'$  is a probability measure: The positivity follows from the fact that  $P$  fulfills the quadrant requirements, and  $P'(\mathbb{R}^n) = 1$  due

to the normalizing factor  $\frac{1}{1-p_M}$  in the definition of  $P'$ . Using Definition 11, one can verify by a short calculation that  $P_M'^{pt} = P$ .

If  $p_M = 1$  then  $P = P_M'^{pt}$  for any probability measure  $P'$ .  $\square$

One challenge for supervisors is to decide whether or not a scenario can be excluded. An answer is given by

*Remark 18.* Theorem 2 together with Theorem 4 is very helpful if one is looking for criteria to exclude scenarios for regulatory purposes while using point mass aggregation: let  $M$  be a scenario set. A subset  $M'$  of  $M$  is called *sufficient* for  $P$  if  $P_{M'}^{pt}$  satisfies the quadrant requirements.

As discussed in section 3, the test whether or not quadrant requirements are fulfilled, is free of subjectivity and reproducible. Because of the equivalence stated in Theorem 2 and 4, these properties carry over to scenarios when using point mass aggregation.

## 6 Further properties of scenario aggregation

We consider the mapping on the set of probability distributions defined by  $P \mapsto P_M^{pt}$ . One might ask whether, knowing  $P_M^{pt}$ , we can reconstruct  $P$ . The answer is provided in

**Theorem 5.** *The mapping defined by  $P \mapsto P_M^{pt}$  is injective for  $p_M < 1$ . The image consists of all probability distributions satisfying the quadrant requirement associated to  $M$  by means of Theorem 4.*

*Proof.* The characterization of the image is a direct consequence of Theorem 4. The injectivity can be seen by observing that

$$P = \frac{1}{1-p_M} \left( P_M^{pt} - \sum_{S \in M} p_S \delta_{d_S} \right),$$

thus we have an explicit formula for  $P$  given  $P_M^{pt}$ .  $\square$

*Remark 19.* Whilst the properties of the mapping induced by point mass aggregation are easily studied, the properties of the corresponding mapping induced by aggregation by shifting do not appear to so clear.

Next, we consider the properties of successive aggregation of scenarios. Here we can see that even point-mass aggregation is not so easily tractable. Consider disjoint scenario sets  $M_1, M_2$  respectively such that  $M_1 \cup M_2$  is

again a scenario set (i.e. total probability of all scenarios  $\leq 1$ ). Then one might ask whether

$$P_{M_1 \cup M_2}, (P_{M_1})_{M_2}, (P_{M_2})_{M_1}$$

all lead to the same result; the same question may be asked for point mass aggregation. The answer is that, even in the case of point mass aggregation, the three terms may generally lead to different results. This is even the case for two different enhanced scenarios, as can be seen from

*Example 20.* Take  $P$  as  $\delta_0$ , the Dirac measure centered at 0, and take  $M_1, M_2$  as sets consisting of one enhanced scenario each, defined by  $(d_1, p_1)$  and  $(d_2, p_2)$ .

Then we have, for point mass aggregation

$$\begin{aligned} P_{M_1 \cup M_2}^{pt} &= (1 - p_1 - p_2)\delta_0 + p_1\delta_{d_1} + p_2\delta_{d_2} \\ (P_{M_1}^{pt})_{M_2} &= (1 - p_1)(1 - p_2)\delta_0 + p_1(1 - p_2)\delta_{d_1} + p_2\delta_{d_2} \\ (P_{M_2}^{pt})_{M_1} &= (1 - p_1)(1 - p_2)\delta_0 + p_1\delta_{d_1} + (1 - p_1)p_2\delta_{d_2} \end{aligned}$$

All three terms generally lead to different results, so it is important when talking about scenario aggregation to specify whether finitely many scenarios are to be aggregated in one step or successively. Furthermore, if scenarios are aggregated successively, the order needs to be specified. To avoid this, we are using the aggregation in one step as a standard in this paper when aggregating a scenario set, which has the advantage that we do not need to specify an order for our scenario sets.

Similar formulas are obtained when using aggregation by shifting.

So a supervisor should be very clear in what he is requiring, because aggregation in one step may well lead to another result than successive aggregation.

## 7 Simulation based approaches and the quadrant definition

During the proof of Theorem 4, we have made use of quadrants consisting of one point only. Whilst theoretically useful, such quadrant requirements lead to difficulties in the context of a simulation based approach. The reason for this is that a simulation generated based on continuous probability distributions will fulfill such a quadrant requirement with probability 0.

For this purpose, we introduce a somewhat restricted definition of quadrants:

**Definition 21.** A quadrant  $Q$  is called *non-degenerate*, if it has non-zero Lebesgue measure.

Therefore, when practically working with quadrant requirements, supervisors should use non-degenerate quadrants if they want to enable companies to apply the usual simulation techniques.

## 8 Scenario aggregation: a more general setting

So far, we discussed two methods, aggregation by shifting and by point mass. In this section we want to outline how these results can be embedded in a more general framework. It turns out, that for theoretical and practical reasons point mass aggregation excels by unique properties within the class of aggregation methods based on mixing. We start with

**Definition 22** ( $\varphi$ -aggregation). Let  $M$  be a scenario set,  $P$  a probability measure on  $(\mathbb{R}^n, \mathfrak{B}_n)$ , and  $(\varphi_S)_S$  an arbitrary family of measurable mappings  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . Define

$$P_M^\varphi = (1 - p_M)P + \sum_{S \in M} p_S \varphi_{S*} P$$

We say that  $P_M^\varphi$  is obtained from  $P$  by  $\varphi$ -aggregating enhanced scenarios from scenario set  $M$  to  $P$ .

The two methods discussed above are easily included, as we can see in

*Example 23.* Point mass aggregation is a  $\varphi$ -aggregation for the family of constant mappings  $\varphi_S : v \mapsto d_S$ , and aggregation by shifting is clearly induced by the family  $\varphi_S := \tau_{d_S}$ .

Example 12 makes only use of *one* property of translations, their feature being *distance preserving* mappings. In order to generalize this counterexample, we need

**Definition 24.** A mapping  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called *expanding*, if  $|x - y| \leq |\varphi(x) - \varphi(y)| \forall x, y \in \mathbb{R}^n$ .

To proceed, we need

**Lemma 3.** An expanding mapping  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  is always injective, and bijective if and only if it is continuous.



*Proof.* We first show injectivity. Let  $x, y \in \mathbb{R}^n$ , and assume  $\varphi(x) = \varphi(y)$ . Then by definition of an expanding map, we have

$$0 = |\varphi(x) - \varphi(y)| \geq |x - y|$$

Thus we have  $|x - y| = 0$  and  $x = y$ .

Now we assume  $\varphi$  to be continuous, so we have to show surjectivity. As  $\mathbb{R}^n$  is connected, it suffices to show that the image  $\varphi(\mathbb{R}^n)$  is both open and closed. First, we note that the image is open due to the open mapping theorem. So it remains to be shown that it is closed. For this, it is sufficient to show that with any convergent sequence  $(y_j)$ ,  $y_j \in \varphi(\mathbb{R}^n)$ , the limit  $y$  is also contained in  $\varphi(\mathbb{R}^n)$ .

Choose  $x_j \in \mathbb{R}^n$  such that  $\varphi(x_j) = y_j$ . By the definition of an expanding map, we have for any  $j, k \in \mathbb{N}$

$$|x_j - x_k| \leq |\varphi(x_j) - \varphi(x_k)| = |y_j - y_k|$$

From this inequality and the fact that  $(y_j)$  is a Cauchy sequence, we conclude that  $(x_j)$  is also a Cauchy sequence. Denote the limit by  $x$ . Then by continuity, we conclude that  $y = \varphi(x)$ , therefore  $y \in \varphi(\mathbb{R}^n)$ , and we have shown the surjectivity of  $\varphi$ .

Conversely, assume surjectivity. Then  $\varphi$  is bijective, so the inverse mapping  $\psi$  can be defined. It is obvious that the definition of an expanding map implies that  $\psi$  is a contraction. Especially,  $\psi$  is Lipschitz-continuous and therefore continuous. Due to the open mapping theorem,  $\psi$  maps open sets to open sets, which means that the inverse image of an open set  $U$  under  $\varphi$  is open, because  $\varphi^{-1}(U) = \psi(U)$ . By definition, this means that  $\varphi$  is continuous.  $\square$

There is a vast literature on expanding maps and on their generalizations on Hilbert spaces other than  $\mathbb{R}^n$ , we refer to ([GR81], [SZ01]). Especially, the generalization of Lemma 3 is only valid under additional assumptions in the case of Hilbert or Banach spaces. The reason why the proof given above cannot be carried over is that the open mapping theorem does not hold in arbitrary Hilbert spaces.

With arguments similar as in example 12 we can now conclude

**Lemma 4.** *Let  $M$  be a scenario set, and  $\varphi_S$  a family of measurable expanding mappings,  $p_{\max} \in [0, 1]$  and  $P$  such that  $P(B) < p_{\max}$  for all balls  $B$  with fixed radius  $R \geq 0$ . Then  $P_M^\varphi$  does not fulfill the quadrant requirement*

$(A, p_{\max})$  for any quadrant  $A$  contained in a ball  $B$  with radius  $R/2$ . Furthermore, if  $\varphi$  is continuous, then  $P$  never fulfills the quadrant requirement  $(A, p_{\max})$  for any quadrant  $A$  contained in a ball  $B$  with radius  $R$ .

Thus, we should exclude the expanding mappings for the purpose of scenario aggregation if we want to be sure that quadrant requirements can be fulfilled. Note that any distance preserving mapping is expanding by definition, so that translations and other isometric mappings are naturally covered by Lemma 4.

Point mass aggregation is induced by constant mappings, which are a special case of contracting mappings  $\varphi$ , where *contracting* is given by the dual notion of expanding, i.e.  $|x - y| \geq |\varphi(x) - \varphi(y)| \forall x, y \in \mathbb{R}^n$ .

*Remark 25.* One might be tempted to try other contracting mapping  $\varphi_S$  than just the constant ones in order to satisfy quadrant requirements by  $\varphi$ -aggregation. In this case, however, we would recall the arguments at the end of section 4. Indeed, it turns out that point mass aggregation is the only one induced by a contracting family without these caveats.

So far, we only used mixing for aggregation of scenarios. One might well ask the question, whether or not there are other algorithms of aggregating them. Suppose there is an arbitrary input distribution  $P$ , and an algorithm unknown to the supervisor (“black box”) which processes  $P$  in finite time into a distribution  $P'$  which satisfies an arbitrary finite set of quadrant requirements. This “black box” is equivalent to the supervisor’s algorithm of point mass aggregation which yields  $P^{pt}$  in finite time:  $P^{pt}$  might well be not equal to  $P'$ , but equivalent in the sense that both  $P'$  and  $P^{pt}$  satisfy the given quadrant requirements. Thus there is always an mixing algorithm which yields equivalent results as the “black box” but is much more transparent then it.

All together, one is well advised not seeking for too long time for an alternative algorithm in order to fulfill quadrant requirements, since it would be anyway equivalent to an aggregation by mixing.

## 9 Quadrant requirements and duality

We have defined a quadrant  $A$  as a finite intersection of affine half-spaces in  $\mathbb{R}^n$ . Now we move from  $\mathbb{R}^n$  to  $\mathfrak{M} := \mathfrak{M}(\mathfrak{B}_n, \mathbb{R})$  the set of finite real-valued Borel-measures (not necessarily positive). For  $\mu_1, \mu_2 \in \mathfrak{M}$  and  $r \in \mathbb{R}$  we set  $(r \cdot \mu_1)(A) := r \cdot \mu_1(A)$  and  $(\mu_1 + \mu_2)(A) := \mu_1(A) + \mu_2(A)$  for all  $A \in \mathfrak{B}_n$ , so  $\mathfrak{M}$  becomes a vector space over  $\mathbb{R}$ . The subset of probability measures is denoted by  $\mathfrak{M}_{prob} := \mathfrak{M}_{prob}(\mathfrak{B}_n, \mathbb{R})$ .

We observe that the map evaluating at  $A$

$$\lambda_A : \mathfrak{M} \rightarrow \mathbb{R}, \mu \mapsto \mu(A)$$

is a continuous linear form on  $\mathfrak{M}$ .

The notion of affine half-space and thus quadrants in definition 1 carries easily over to arbitrary (not necessarily finite dimensional) real vector spaces, and the notion of  $\mu$  satisfying a quadrant requirement can easily be generalized from probability measures to any real-valued measure:  $\mu$  satisfies the quadrant requirement  $(A, p)$  if  $\mu(A) \geq p$ .

**Definition 26.** We say that  $\mu$  belongs to the *acceptance set* of quadrant requirement  $(A, p)$  if  $\mu$  satisfies  $(A, p)$ ; we use the same terminology in case of an arbitrary set of quadrant requirements.

The condition for belonging to the acceptance set of a single quadrant requirement can be rewritten as follows

$$\lambda_A(\mu) \geq p$$

From this, we immediately obtain the following

**Lemma 5.** *The acceptance set of a single quadrant requirement is an affine half-space in  $\mathfrak{M}$ .*

Therefore, while a quadrant is a finite intersection of affine half-spaces in  $\mathbb{R}^n$ , the acceptance sets of a finite set of quadrant requirements is an intersection of finitely many affine half-spaces in  $\mathfrak{M}$ .

**Theorem 6.** *The acceptance set of an arbitrary set of quadrant requirements is closed and convex, both in  $\mathfrak{M}_{prob}$  and in  $\mathfrak{M}$ .*

*Proof.* In the case of  $\mathfrak{M}$ , the assertion follows from the fact that due to Lemma 5 any acceptance set is an intersection of affine half-spaces. To extend it to  $\mathfrak{M}_{prob}$ , we note that the condition for a real-valued measure to be a probability measure can also be written as follows as an intersection of half-spaces:

$$\forall A \in \mathfrak{B}_n : \lambda_A(\mu) \geq 0, \lambda_{\mathbb{R}^n}(\mu) = 1$$

Hence also in this case, the acceptance sets are intersections of affine half-spaces and therefore convex.  $\square$

*Remark 27.* Theorem 6 means that a supervisor may only use quadrant requirements to describe its regulatory requirements if the set of distributions which is accepted by the supervisor is closed and convex.

Conversely, one may ask the question whether any convex set of measures can be written as acceptance set of a set of quadrant requirements. For this purpose, we need to study the dual space of  $\mathfrak{M}$ , and it turns out that it is too rich in the sense that not all closed convex sets may be described as quadrant requirements in the sense of our definition. However, we can generalize the notion of quadrant requirement as follows to obtain a result.

**Definition 28.** Let  $\mu \in \mathfrak{M}$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  a function such that its  $\mu$ -integral exists, and  $p \in \mathbb{R}$ . A *generalized requirement* defined by  $(g, p)$  is a condition of the form

$$\int g d\mu \geq p$$

and  $\mu$  is said to belong to the acceptance set of  $(g, p)$ ; we use similar terminology in the case of a set of generalized requirements.

*Remark 29.* One can easily see that quadrant requirements are a special case, by taking  $g = \chi_A$  to be the characteristic function of a quadrant  $A$ .

**Theorem 7.** *Any closed convex subset of  $\mathfrak{M}$  can be described as acceptance set of a set of generalized requirements.*

*Proof.* It is a known fact that the dual space of  $\mathfrak{M}$  can be identified with  $L^\infty(\mathbb{R}^n)$  (with respect to the Lebesgue measure), hence any linear form on  $\mathfrak{M}$  can be expressed by integrating over a function  $g \in L^\infty(\mathbb{R}^n)$  (cf. [LA93], Chapter VII, Theorem 2.2 in conjunction with Corollary 4.3).

Similarly, by the Hahn-Banach separation theorem (cf. [LA93], Appendix to Chapter IV, Theorem 1.2), any convex subset of a Banach space can be expressed as an intersection of affine half-spaces.

Taking these two facts together, the theorem follows immediately.  $\square$

*Remark 30.* We see that a supervisor may have to use generalized requirements in case the acceptance set is more complicated in the sense that inspection of weights on (possibly infinitely many) quadrants is not sufficient to decide whether a distribution is accepted or not. In this case, as a first step between quadrant requirements and arbitrary generalized requirements, it might be useful to try whether it is possible to get along with considering generalized requirements defined by step functions  $g$  instead of characteristic functions of quadrants.

## 10 Conclusion

We introduced and discussed scenarios and different methods for aggregating them at risk factor level. We defined quadrant requirements and investigated their relationship and compatibility with different scenario aggregation methods. It turns out that aggregation methods based on contractive mappings, and especially point mass aggregation, play an important role in this respect. Furthermore, we studied generalized requirements and showed that they may be used to describe any closed convex acceptance set of risk factor distributions.

## A Appendix: Proof of Lemma 2

Let  $S \subseteq \mathbb{R}^n$ . We define the asymptotic cone of  $S$ , denoted by  $\text{Cone}(S)$ . It consists of all  $x \in \mathbb{R}^n$  such that  $y + tx \in S$  for all  $t > 0$  and  $y \in S$ . Some more background of the asymptotic cone together with some useful properties can be found in [HL01], p. 39.

Initially, we note that, for a fixed dimension  $n$ , it suffices to show the lemma for *closed* convex sets. Indeed, assume the lemma is true in this case, and let  $S$  be an arbitrary convex set.

$\bar{S}$  is closed and convex, hence by assumption, we can apply the lemma to  $\bar{S}$ . Therefore,  $\bar{S}$  is either two-sided constrained, or it contains arbitrarily large balls. In case  $\bar{S}$  is two-sided constrained, then so is  $S$ , so we are done. Otherwise,  $\bar{S}$  contains balls  $B_{2R}$  of radius  $2R$  for any  $R > 0$ . We claim that the corresponding smaller ball  $B_R$  of radius  $R$  around the same point is then contained in  $S$ . Indeed, we can find finitely many points  $z_j \in B_{2R}$  such that the convex ball of radius  $2R$  is contained in the convex hull of these points. For each point, we can find  $z'_j \in S$  such that  $|z_j - z'_j| < \epsilon$ . By choosing  $\epsilon > 0$  sufficiently small,  $B_R$  is also contained in the convex hull of the points  $z'_j$ , which itself is a subset of  $S$  by convexity. Hence  $S$  contains a ball of radius  $R$ .

Next, we proceed to show the lemma in dimension  $n$ , and from what we have seen above, we may assume that  $S$  is closed. We proceed by induction on  $n$ , the start of the induction is obvious and has been described in Example 15. So we can assume that the lemma is true in all dimensions  $< n$ , and for arbitrary convex sets (not necessarily closed).

We choose a maximal linear independent subset of  $\text{Cone}(S)$ , denoted by  $x_1, \dots, x_k$ .

If  $k = n$ , we are finished, because  $S$  then contains arbitrarily large,

non-degenerate  $n$ -simplices, and thus arbitrarily large balls.

Assume  $k < n$ . Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$  be a linear map such that the kernel is the subspace generated by  $v_1, \dots, v_k$ . Define  $S' = L(S)$ .  $S'$  is also a convex set.

If  $S'$  is two-sided constrained, then so is  $S$ , because if  $\lambda$  is a non-zero linear form on  $\mathbb{R}^{n-k}$  such that  $\lambda(S')$  is bounded, then  $\lambda' := \lambda \circ L$  is a non-zero linear form on  $\mathbb{R}^n$  such that  $\lambda'(S)$  is bounded.

Hence we can assume  $S'$  is not two-sided constrained. As  $S'$  lies in a vector space of dimension  $< n$ , we can apply the lemma to  $S'$  by induction assumption, concluding that  $S'$  contains arbitrarily large balls, and thus an arbitrarily large  $n-k$  cube spanned by points  $y_1, \dots, y_l \in S'$ . Assume  $x_j \in S$  is mapped to  $y_j$  under  $L$ . Then by the invariance of the asymptotic cone (cf. Proposition 2.2.1 in [HL01]; note that we are now making use of the fact that  $S$  is closed) we see that  $S$  contains a one-sided cylinder over the cube spanned by the points  $x_1, \dots, x_l$  and the directions  $v_1, \dots, v_k$ . As we can make the cube arbitrarily large, it is easily verified that this cylinder, and thus  $S$ , contains arbitrarily large  $n$ -cubes, and thus arbitrarily large balls.

For elaborating the last step, we have make use of the fact that by the equivalence of norms on  $\mathbb{R}^n$ , the property of containing large balls (or cubes) is preserved under any linear isomorphism, and we may thus without loss of generality assume that the  $v_k$  are the standard unit vectors, in which case the assertion follows immediately.

$\bar{S}$  is closed and convex, hence from what we have just proved,  $\bar{S}$  is either two-sided constrained, or it contains arbitrarily large balls. In case  $\bar{S}$  is two-sided constrained, then so is  $S$ , so we are done. Otherwise,  $\bar{S}$  contains balls  $B_{2R}$  of radius  $2R$  for any  $R > 0$ . We claim that the corresponding smaller ball  $B_R$  of radius  $R$  around the same point is then contained in  $S$ . Indeed, we can find finitely many points  $z_j \in B_{2R}$  such that the convex ball of radius  $2R$  is contained in the convex hull of these points. for each point, we can find  $z'_j \in S$  such that  $|z_j - z'_j| < \epsilon$ . By choosing  $\epsilon > 0$  sufficiently small,  $B_R$  is also contained in the convex hull of the points  $z'_j$ , which itself is a subset of  $S$  by convexity. Hence  $S$  contains a ball of radius  $R$ .

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